Univalence implies Function Extensionality

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Abstract

Assume a judgemental η-rule for functions and univalence. Given $f:\Pi(a:A),B$, let: $\mathsf{himg}_f(a) :\equiv (\Sigma(b:Ba), fa = b)$ for $a:A$, $\mathsf{hgraph}(f) :\equiv (\Sigma(a:A), \mathsf{himg}_f(a))$, and $\mathsf{pr}_A : \mathsf{hgraph}(f) \to A$ be the canonical projection. Note that pr_A is an equivalence (f gives the data). Construct a retract $r:\mathsf{fib}_{\mathsf{pr}_A \circ(-)}(\mathsf{id}_A) \to (\Sigma(g:\Pi(a:A),B), f \sim g)$ as $r(k,p) :\equiv (\mathsf{fst} \circ \mathsf{im}, \mathsf{snd} \circ \mathsf{im})$ where $\mathsf{im} :\equiv \lambda a.\, \mathsf{tr}_{A,\mathsf{himg}}(pa, \mathsf{snd}(ka))$, as witnessed by $s(g,h) :\equiv (\lambda a.(a,(ga,ha)), \mathsf{refl}_{\mathsf{id}_A})$. Using univalence, deduce that the function $\mathsf{pr}_A \circ(-):(A \to \mathsf{hgraph}(f)) \to (A \to A)$ is an equivalence so that $\Sigma(g:\Pi(a:A),B), f \sim g$ is contractible and a set. Observe $\mathsf{tr}_{\Pi(a:A),B,f\sim}(f,g,p,\lambda a.\,\mathsf{refl}_{fa}) = \mathsf{happly}(p)$, and conclude that $\mathsf{fib}_{\mathsf{happly}}(h) \equiv (\Sigma(p:f=g), \mathsf{happly}(p) = h) \simeq ((f, \lambda a.\,\mathsf{refl}_{fa}) = (g,h))$ is contractible. ■

1. Why this proof?

Given a function $f:\Pi(a:A),B$ and $a:A$, we know that the homotopy image of a, defined as $\mathsf{himg}_f(a) :\equiv \Sigma(b:Ba), fa = b$, is contractible. Should we define the corresponding homotopy graph of f as $\mathsf{hgraph}(f) :\equiv \Sigma(a:A), \mathsf{himg}_f(a)$, the previous observation allows us to prove that $\mathsf{pr}_A : \mathsf{hgraph}(f) \to A$ is an equivalence.

While not interesting on its own, we may note that functions $g:\Pi(a:A),B$ which are homotopic to f are *nearly* the same things as functions $A \to \mathsf{hgraph}(f)$. Nearly only so because that type has things which are 'wild', they may map a term a to a term $b':B(a')$ in a fibre not necessarily related to $B(a)$ and so may not be readily rectified into functions. Some meditation on this reveals that terms $\overline{g}:A \to \mathsf{hgraph}(f)$ corresponding to functions $g:\Pi(a:A),B$ and witnesses $f \sim g$ are precisely those for which we have a proof $\mathsf{pr}_A \circ \overline{g} \sim \mathsf{id}_A$.

Thus we see that the data carried by the type $\Sigma(g:\Pi(a:A),B), f \sim g$ may be equivalently found in some sub-type of $A \to \mathsf{hgraph}(f)$. Were we able to show that this sub-type was contractible, we would in particular show that if $f \sim g$ then $f = g$. Unfortunately, in the absence of function extensionality, there does not appear to be an easy way to pin-down this sub-type of functions in such a way that contractibility is easily shown.

Instead, we restrict our attention to an *a priori* 'larger' sub-type of $A \to \mathsf{hgraph}(f)$, viz., those terms $\overline{g}$ for which we have a proof $p : \mathsf{pr}_A \circ g = \mathsf{id}_A$. As the existence of such a term p certainly entails the existence of a term $\mathsf{happly}(p): \mathsf{pr}_A \circ g \sim \mathsf{id}_A$ we expect (and may prove) that this sub-type retracts onto $\Sigma(g:\Pi(a:A),B), f \sim g$.